Lecture 6 - Sep 23

Math Review

Constructing All Relations
Domain, Range, Inverse
Image, Restrictions, Subtractions

Announcements/Reminders

- Today's class: notes template posted
- Event-B Summary Document
- Priorities:
  - \+ Lab1 → Review
  - \+ Lab2 → Due: This Tuesday (Sep 23)
- Released:
  - \+ ProgTest guide
  - \+ 2 Practice Tests
  - \+ Lab1 solution

Cardinality of Power Set: Interpreting Formula

- Calculate by considering subsets of various cardinalities.
- Calculate by considering whether a member should be included.

Want to know : $|\mathbb{P}(S)|$

$$|\mathbb{P}(S)| = \underbrace{\overset{1}{\binom{|S|}{0}}}_{\phi} + \binom{|S|}{1} + \binom{|S|}{2} + \cdots + \underbrace{\overset{1}{\binom{|S|}{|S|}}}_{S}$$

$$|\mathbb{P}(S)| = \underbrace{2 \times 2 \times \cdots \times 2}_{|S|} = 2^{|S|}$$

$S = \{a, b, c\}$

$|S| = 3$, $2^{3} = 8$

exclude c

include c

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ & \vdots & \\ 1 & 1 & 1 \end{pmatrix} \begin{array}{l} \longrightarrow \phi \\ \longrightarrow \{c\} \\ \\ \\ \longrightarrow \{a, b, c\} \end{array}$$

relation on S and S

acceptable

e.g. id

Relation : set of ordered pairs

tuples.

e.g. a relation on $S \, \boxed{\{1, 2, 3\}}$ and $T \, \boxed{\{a, b\}}$

$\{(x, y) \mid x \in S \wedge y \in T\}$

- Is $(1, a)$ a relation on S and T?

  No! ∵ $(1, a)$ is not a set.

- Is $\{(1, a)\}$ a relation on S and T? YES

- Is $\{(b, 2)\}$ a relation? No. order is wrong! ($b \notin S$, $2 \notin T$)

$R_1 = \{(1, a), (3, b)\}$

$R_2 = \{(3, b), (1, a)\}$

$R_1 = R_2$

What is the min relation on S and T? $\emptyset$

What is the max relation on S and T? $S \times T$

Set of Possible Relations

subsets of max relation

max relation

- **Set** of possible **relations** on S and T: $\mathbb{P}(\boxed{S \times T})$
- Dedicated symbol for **set** of possible **relations** on S and T: $S \leftrightarrow T$
- Declare that set r is a **relation** on S and T: $r \in \mathbb{P}(S \times T)$ $\quad r \in S \leftrightarrow T$

Example: Enumerate all relations on $\overbrace{\{a, b\}}^{S}$ and $\overbrace{\{2, 4\}}^{T}$.

Hint: How many? $2^{|S \times T|} = 2^{2 \times 2} = 2^4 = 16$

max rel: $\boxed{\{(a,2), (a,4), (b,2), (b,4)\}}$ $S \times T$

$|S \times T| = 4$

$\emptyset$ relation of card. 0 $\binom{4}{0} = 1$

$\{(a,2)\}, \{(a,4)\}, \{(b,2)\}, \{(b,4)\}$ → relations of card 1 $\binom{4}{1}$

→ relations of card 2 $\binom{4}{2}$

→ relations of card. 3 $\binom{4}{3}$

$\{(a,2), (a,4), (b,2), (b,4)\}$ relation of card. 4 $\binom{4}{4}$

$* \{ r \mid r \in Dep. \leftrightarrow Des \wedge |r| = 2 \}$

Exercise

$$Departure = \{toronto, montreal, vancouver\}^{3}$$

$$Destination = \{beijing, seoul, penang\}^{3}$$

$airline \in \boxed{Departure \leftrightarrow Destination}$ $\mathbb{P}(Dep. \times Des.)$

a single relation

$$\binom{\boxed{|Dep. \times Des|}^{9}}{2} = 36 \quad (1) \quad |Dep. \leftrightarrow Des.| = 2^{|Dep \times Des|} = 2^{3 \times 3} = 2^9 = \underline{\underline{512}}$$

(2) enumerate * $\boxed{\text{those relations with card 2.}}$

$$S \longleftrightarrow T = \mathbb{P}(S \times T)$$

$r \in$ Alphabet $\leftrightarrow \mathbb{Z}$

Relational Operations: Domain, Range, $\boxed{\text{Inverse}}$

r = {(a, 1), (b, 2), (c, 3), (a, 4), (b, 5), (c, 6), (d, 1), (e, 2), (f, 3)}

$dom(r) = \{a, b, c, d, e, f\}$ $\quad dom(r) \subseteq$ Alphabet

r = {(a, 1), (b, 2), (c, 3), (a, 4), (b, 5), (c, 6), (d, 1), (e, 2), (f, 3)}

$ran(r) = \{1, 2, 3, 4, 5, 6\}$ $\quad ran(r) \subseteq \mathbb{Z}$

$r^\sim$

r = {(a, 1), (b, 2), (c, 3), (a, 4), (b, 5), (c, 6), (d, 1), (e, 2), (f, 3)}

$r^\sim = \{(1, a), (2, b), (3, c), (4, a), (5, b), (6, c), (1, d), (2, e), (3, f)\}$

Exercise: Relate the domains and ranges of $\boxed{r}$ and $\boxed{\text{its inverse}}$.

$r^\sim$

$$\left.\begin{aligned} dom(r) &= ran(r^\sim) \\ ran(r) &= dom(r^\sim) \end{aligned}\right)$$

algebraic properties.

Relational Operations: Image

$r \in \text{Alphabet} \leftrightarrow \mathbb{Z}$

$r[\{a,h\}] = \underbrace{r[\{a\}]}_{\{1,4\}} \cup \underbrace{r[\{h\}]}_{\varnothing}$

$r = \{(a, 1), (b, 2), (c, 3), (a, 4), (b, 5), (c, 6), (d, 1), (e, 2), (f, 3)\}$

$r[\underbrace{\{a,b\}}_{S}] = \{ r' \mid (d, r') \in r \wedge d \in \{a,b\} \} = \underbrace{\{1, 2, 4, 5\}}_{\subseteq \, ran(r)}$

$S \subseteq \text{Alphabet}$

$S \subseteq dom(r)$ ✗ not necessary.

$r[\underbrace{\{g\}}_{\subseteq \text{Alphabet}}] = \varnothing$ no value mapped from g in r.

Exercises

- Image of {a, b} on r?
- Image of {1, 2} on r? $r[\{1,2\}]$ undefined! $\{1,2\} \not\subseteq \text{Alphabet}$
- Image of {1, 2} on the inverse of r? $r^{\sim}[\{1,2\}] = \{a, b, d, e\}$
- Calculate r's range via an image. $r[dom(r)] = ran(r)$
- Calculate r's domain via an image. $r^{\sim}[\underbrace{ran(r)}_{dom(r^{\sim})}] = dom(r)$

$r \in S \leftrightarrow T$

$s \subseteq S \qquad t \subseteq T$

a new relation.

* $s \lhd r = \{ (d, r') \mid (d, r') \in r \wedge d \in s \}$

	domain	range
Restriction	* $\boxed{s \lhd r}$	$r \rhd t$
Subtraction	$s \mathbin{⩤} r$	$r \mathbin{⩥} t$

Each of these operators returns a new relation.

$s = \{a, b\}$

Relational Operations: Restrictions vs. Subtractions

$r = \{(a, 1), (b, 2), (c, 3), (a, 4), (b, 5), (c, 6), (d, 1), (e, 2), (f, 3)\}$

$\{a, b\} \lhd r = \{(a, 1), (b, 2), (a, 4), (b, 5)\}$

$r = \{\cancel{(a, 1)}, \cancel{(b, 2)}, (c, 3), \cancel{(a, 4)}, \cancel{(b, 5)}, (c, 6), (d, 1), (e, 2), (f, 3)\}$

$\{a, b\} ⩤ r = \{(c, 3), (c, 6), (d, 1), (e, 2), (f, 3)\}$

$r = \{(a, 1), (b, 2), (c, 3), (a, 4), (b, 5), (c, 6), (d, 1), (e, 2), (f, 3)\}$

$r \rhd \{1, 2\} = \{(a, 1), (b, 2), (d, 1), (e, 2)\}$

$r = \{\cancel{(a, 1)}, \cancel{(b, 2)}, (c, 3), (a, 4), (b, 5), (c, 6), \cancel{(d, 1)}, \cancel{(e, 2)}, (f, 3)\}$

$r ⩥ \{1, 2\} = \{(c, 3), (a, 4), (b, 5), (c, 6), (f, 3)\}$

Relational Operations: Overriding

r = {(a, 1), (b, 2), (c, 3), (a, 4), (b, 5), (c, 6), (d, 1), (e, 2), (f, 3)}

Example: Calculate r **overridden with** {(a, 3), (c, 4)}

Hint: Decompose results to those **in** t's domain and those **not in** t's domain.

$$r \Leftarrow t = \{(d, r') \mid (d, r') \in t \lor ((d, r') \in r \land d \notin dom(t))\}$$

relation relation

$$= \{(d, r') \mid (d, r') \in t\} \cup \{(d, r') \mid (d, r') \in r \land d \notin dom(t)\}$$